Warm Up

Simplify.

1. $(2^6)(2^8)$ 2^{14}

2. $(3^{-2})(3^5)$ 3^3

3. $\frac{3^{12}}{3^4}$ 3^8

4. $\frac{4^3}{4^{-1}}$ 4^4

5. $(7^3)^5$ 7^{15}

Write in exponential form.

6. $\log_x x = 1$ $x^1 = x$

7. $0 = \log_x 1$ $x^0 = 1$

Objectives

Use properties to simplify logarithmic expressions.

Translate between logarithms in any base.

The logarithmic function for pH that you saw in the previous lessons, $pH = -\log[H^+]$, can also be expressed in exponential form, as $10^{-pH} = [H^+]$.

Because logarithms are exponents, you can derive the properties of logarithms from the properties of exponents

Remember that to *multiply* powers with the same base, you *add* exponents.

$$b^m b^n = b^{m+n}$$

Product Property of Logarithms

For any positive numbers m, n, and b $(b \neq 1)$,

WORDS	NUMBERS	ALGEBRA
The logarithm of a product is equal to the sum of the logarithms of its factors.	$\log_3 1000 = \log_3(10 \cdot 100)$ $= \log_3 10 + \log_3 100$	$\log_b mn = \log_b m + \log_b n$

The property in the previous slide can be used in reverse to write a sum of logarithms (exponents) as a single logarithm, which can often be simplified.

Helpful Hint

Think: $\log_j + \log_a + \log_m = \log_{jam}$

Example 1: Adding Logarithms

Express $\log_6 4 + \log_6 9$ as a single logarithm. Simplify.

$\log_6 4 + \log_6 9$

$\log_6 (4 \bullet 9)$ — *To add the logarithms, multiply the numbers.*

$\log_6 36$ — *Simplify.*

2 — *Think: $6^? = 36$.*

Check It Out! Example 1a

Express as a single logarithm. Simplify, if possible.

$\log_5 625 + \log_5 25$

$\log_5 (625 \bullet 25)$ — *To add the logarithms, multiply the numbers.*

$\log_5 15{,}625$ — *Simplify.*

6 — *Think: $5^? = 15625$*

Check It Out! Example 1b

Express as a single logarithm. Simplify, if possible.

$\log_{\frac{1}{3}} 27 + \log_{\frac{1}{3}} \frac{1}{9}$

$\log_{\frac{1}{3}} (27 \bullet \frac{1}{9})$ — *To add the logarithms, multiply the numbers.*

$\log_{\frac{1}{3}} 3$ — *Simplify.*

-1 — *Think: $\frac{1}{3}^{?} = 3$*

Remember that to *divide* powers with the same base, you *subtract* exponents

$$\frac{b^m}{b^n} = b^{m-n}$$

Because logarithms are exponents, subtracting logarithms with the same base is the same as finding the logarithms of the quotient with that base.

Quotient Property of Logarithms

For any positive numbers m, n, and b $(b \neq 1)$,

WORDS	NUMBERS	ALGEBRA
The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.	$\log_5\left(\frac{16}{2}\right) = \log_5 16 - \log_5 2$	$\log_b \frac{m}{n} = \log_b m - \log_b n$

The property above can also be used in reverse.

Caution

Just as a^5b^3 cannot be simplified, logarithms must have the same base to be simplified.

Example 2: Subtracting Logarithms

Express $\log_5 100 - \log_5 4$ as a single logarithm. Simplify, if possible.

$\log_5 100 - \log_5 4$

$\log_5(100 \div 4)$ — *To subtract the logarithms, divide the numbers.*

$\log_5 25$ — *Simplify.*

2 — *Think: $5^? = 25$.*

Check It Out! Example 2

Express $\log_7 49 - \log_7 7$ as a single logarithm. Simplify, if possible.

$\log_7 49 - \log_7 7$

$\log_7 (49 \div 7)$ — *To subtract the logarithms, divide the numbers*

$\log_7 7$ — *Simplify.*

1 — *Think: $7^? = 7$.*

Because you can multiply logarithms, you can also take powers of logarithms.

Power Property of Logarithms

For any real number p and positive numbers a and b $(b \neq 1)$,

WORDS	NUMBERS	ALGEBRA
The logarithm of a power is the product of the exponent and the logarithm of the base.	$\log 10^3$ $\log(10 \cdot 10 \cdot 10)$ $\log 10 + \log 10 + \log 10$ $3 \log 10$	$\log_b a^p = p \log_b a$

Example 3: Simplifying Logarithms with Exponents

Express as a product. Simplify, if possible.

A. $\log_2 32^6$

$6\log_2 32$

$6(5) = 30$

Because $2^5 = 32$, $\log_2 32 = 5$.

B. $\log_8 4^{20}$

$20\log_8 4$

$20\left(\frac{2}{3}\right) = \frac{40}{3}$

Because $8^{\frac{2}{3}} = 4$, $\log_8 4 = \frac{2}{3}$.

Check It Out! Example 3

Express as a product. Simplify, if possibly.

a. $\log 10^4$

$4\log 10$

$4(1) = 4$

Because $10^1 = 10$, $\log 10 = 1$.

b. $\log_5 25^2$

$2\log_5 25$

$2(2) = 4$

Because $5^2 = 25$, $\log_5 25 = 2$.

Check It Out! Example 3

Express as a product. Simplify, if possibly.

c. $\log_2 \left(\frac{1}{2}\right)^5$

$5\log_2 \left(\frac{1}{2}\right)$

$5(-1) = -5$

Because $2^{-1} = \frac{1}{2}$, $\log_2 \frac{1}{2} = -1$.

Exponential and logarithmic operations undo each other since they are inverse operations.

Inverse Properties of Logarithms and Exponents

For any base b such that $b > 0$ and $b \neq 1$,

ALGEBRA	EXAMPLE
$\log_b b^x = x$	$\log_{10} 10^7 = 7$
$b^{\log_b x} = x$	$10^{\log_{10} 2} = 2$

Example 4: Recognizing Inverses

Simplify each expression.

a. $\log_3 3^{11}$

$\log_3 3^{11}$

11

b. $\log_3 81$

$\log_3 3 \cdot 3 \cdot 3 \cdot 3$

$\log_3 3^4$

4

c. $5^{\log_5 10}$

$5^{\log_5 10}$

10

Check It Out! Example 4

a. Simplify $\log 10^{0.9}$

$\log 10^{0.9}$

0.9

b. Simplify $2^{\log_2(8x)}$

$2^{\log_2(8x)}$

$8x$

Most calculators calculate logarithms only in base 10 or base *e* (see Lesson 7-6). You can change a logarithm in one base to a logarithm in another base with the following formula.

Change of Base Formula

For $a > 0$ and $a \neq 1$ and any base b such that $b > 0$ and $b \neq 1$,

ALGEBRA	EXAMPLE
$\log_b x = \dfrac{\log_a x}{\log_a b}$	$\log_4 8 = \dfrac{\log_2 8}{\log_2 4}$

Example 5: Changing the Base of a Logarithm

Evaluate $\log_{32}8$.

Method 1 Change to base 10

$$\log_{32}8 = \frac{\log 8}{\log 32}$$

$$\approx \frac{0.903}{1.51}$$ *Use a calculator.*

$$\approx 0.6$$ *Divide.*

Example 5 Continued

Evaluate $\log_{32}8$.

Method 2 Change to base 2, because both 32 and 8 are powers of 2.

$$\log_{32}8 = \frac{\log_2 8}{\log_2 32} = \frac{3}{5}$$

Use a calculator.

$$= 0.6$$

Check It Out! Example 5a

Evaluate $\log_9 27$.

Method 1 Change to base 10.

$$\log_9 27 = \frac{\log 27}{\log 9}$$

$$\approx \frac{1.431}{0.954}$$ *Use a calculator.*

$$\approx 1.5$$ *Divide.*

Check It Out! Example 5a Continued

Evaluate $\log_9 27$.

Method 2 Change to base 3, because both 27 and 9 are powers of 3.

$$\log_9 27 = \frac{\log_3 27}{\log_3 9} = \frac{3}{2}$$

Use a calculator.

$$= 1.5$$

Check It Out! Example 5b

Evaluate $\log_8 16$.

Method 1 Change to base 10.

$$\text{Log}_8 16 = \frac{\log 16}{\log 8}$$

$$\approx \frac{1.204}{0.903}$$ *Use a calculator.*

$$\approx 1.3$$ *Divide.*

Check It Out! Example 5b Continued

Evaluate $\log_8 16$.

Method 2 Change to base 4, because both 16 and 8 are powers of 2.

$$\log_8 16 = \frac{\log_4 16}{\log_4 8} = \frac{2}{1.5}$$ *Use a calculator.*

$$= 1.3$$

Logarithmic scales are useful for measuring quantities that have a very wide range of values, such as the intensity (loudness) of a sound or the energy released by an earthquake.

Helpful Hint

The Richter scale is logarithmic, so an increase of 1 corresponds to a release of 10 times as much energy.

Example 6: Geology Application

The tsunami that devastated parts of Asia in December 2004 was spawned by an earthquake with magnitude 9.3 How many times as much energy did this earthquake release compared to the 6.9-magnitude earthquake that struck San Francisco in1989?

$$M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$

The Richter magnitude of an earthquake, M, is related to the energy released in ergs E given by the formula.

$$9.3 = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$

Substitute 9.3 for M.

Example 6 Continued

$$\left(\frac{3}{2}\right)9.3 = \left(\frac{3}{2}\right)\frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$

Multiply both sides by $\frac{3}{2}$.

$$13.95 = \log\left(\frac{E}{10^{11.8}}\right)$$

Simplify.

$$13.95 = \log E - \log 10^{11.8}$$

Apply the Quotient Property of Logarithms.

$$13.95 = \log E - 11.8$$

Apply the Inverse Properties of Logarithms and Exponents.

Example 6 Continued

$25.75 = \log E$ — *Given the definition of a logarithm, the logarithm is the exponent.*

$10^{25.75} = E$

$5.6 \times 10^{25} = E$ — *Use a calculator to evaluate.*

The magnitude of the tsunami was 5.6×10^{25} ergs.

Example 6 Continued

$$M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$

$$6.9 = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$

Substitute 6.9 for M.

$$\left(\frac{3}{2}\right)6.9 = \left(\frac{3}{2}\right)\frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$

Multiply both sides by $\frac{3}{2}$ *.*

$$10.35 = \log\left(\frac{E}{10^{11.8}}\right)$$

Simplify.

$$10.35 = \log E - \log 10^{11.8}$$

Apply the Quotient Property of Logarithms.

Example 6 Continued

$10^{22.15} = E$ *Apply the Inverse Properties of Logarithms and Exponents.*

$22.15 = \log E$ *Given the definition of a logarithm, the logarithm is the exponent.*

$1.4 \times 10^{22} = E$ *Use a calculator to evaluate.*

The magnitude of the San Francisco earthquake was 1.4×10^{22} ergs.

The tsunami released $\frac{5.6 \times 10^{25}}{1.4 \times 10^{22}}$ = 4000 times as much energy as the earthquake in San Francisco.

Check It Out! Example 6

How many times as much energy is released by an earthquake with magnitude of 9.2 by an earthquake with a magnitude of 8?

$$M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$

$$9.2 = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$ *Substitute 9.2 for M.*

$$\left(\frac{3}{2}\right)9.2 = \left(\frac{3}{2}\right)\frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$ *Multiply both sides by $\frac{3}{2}$.*

$$13.8 = \log\left(\frac{E}{10^{11.8}}\right)$$ *Simplify.*

Check It Out! Example 6 Continued

$13.8 = \log E - \log 10^{11.8}$ *Apply the Quotient Property of Logarithms.*

$13.8 = \log E - 11.8$ *Apply the Inverse Properties of Logarithms and Exponents.*

$25.6 = \log E$ *Given the definition of a logarithm, the logarithm is the exponent.*

$10^{25.6} = E$

$4.0 \times 10^{25} = E$ *Use a calculator to evaluate.*

The magnitude of the earthquake is 4.0×10^{25} ergs.

Check It Out! Example 6 Continued

$$M = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$

$$8.0 = \frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$ *Substitute 8.0 for M.*

$$\left(\frac{3}{2}\right)8.0 = \left(\frac{3}{2}\right)\frac{2}{3}\log\left(\frac{E}{10^{11.8}}\right)$$ *Multiply both sides by $\frac{3}{2}$.*

$$12 = \log\left(\frac{E}{10^{11.8}}\right)$$ *Simplify.*

Check It Out! Example 6 Continued

$$12 = \log E - \log 10^{11.8}$$

Apply the Quotient Property of Logarithms.

$$12 = \log E - 11.8$$

Apply the Inverse Properties of Logarithms and Exponents.

$$23.8 = \log E$$

Given the definition of a logarithm, the logarithm is the exponent.

$$10^{23.8} = E$$

$$6.3 \times 10^{23} = E$$

Use a calculator to evaluate.

Check It Out! Example 6 Continued

The magnitude of the second earthquake was 6.3×10^{23} ergs.

The earthquake with a magnitude 9.2 released was $\frac{4.0 \times 10^{25}}{6.3 \times 10^{23}} \approx 63$ times greater.

Lesson Quiz: Part I

Express each as a single logarithm.

1. $\log_6 9 + \log_6 24$ — $\log_6 216 = 3$

2. $\log_3 108 - \log_3 4$ — $\log_3 27 = 3$

Simplify.

3. $\log_2 8^{10,000}$ — $30,000$

4. $\log_4 4^{x-1}$ — $x - 1$

5. $10^{\log 125}$ — 125

6. $\log_{64} 128$ — $\frac{7}{6}$

Lesson Quiz: Part II

Use a calculator to find each logarithm to the nearest thousandth.

7. $\log_3 20$ 2.727

8. $\log_{\frac{1}{2}} 10$ –3.322

9. How many times as much energy is released by a magnitude-8.5 earthquake as a magntitude-6.5 earthquake?

1000