14.1 Graphing Sine, Cosine, and Tangent Functions

What *you should learn*

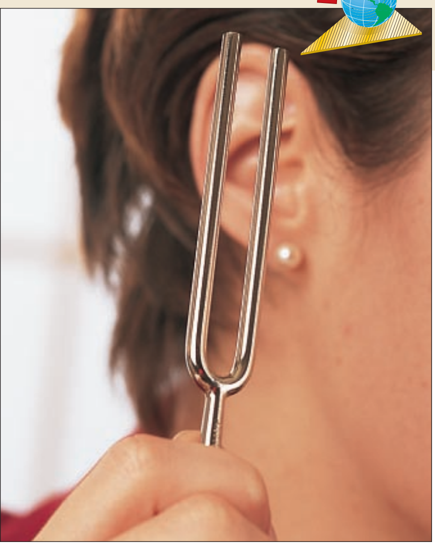
GOAL 1 Graph sine and cosine functions, as applied in **Example 3**.

GOAL 2 Graph tangent functions.

Why *you should learn it*

▼ To model repeating **real-life** patterns, such as the vibrations of a tuning fork in **Ex. 52**.

REAL LIFE

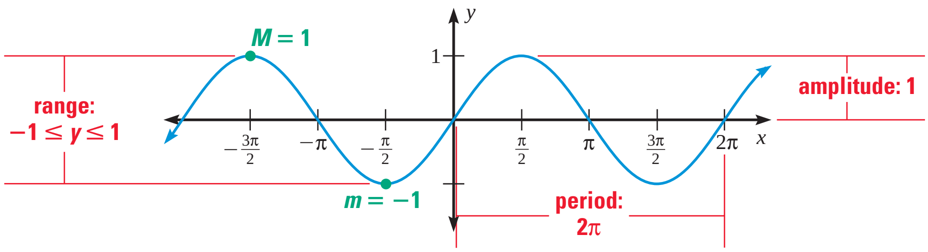


GOAL 1 GRAPHING SINE AND COSINE FUNCTIONS

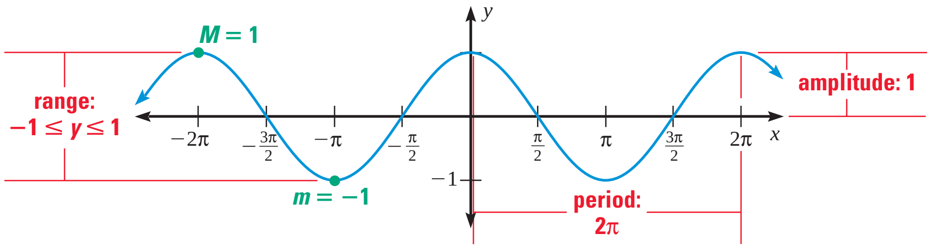
In this lesson you will learn to graph functions of the form $y = a \sin bx$ and $y = a \cos bx$ where a and b are positive constants and x is in radian measure. The graphs of all sine and cosine functions are related to the graphs of

$$y = \sin x \qquad \text{and} \qquad y = \cos x$$

which are shown below.



Graph of y = sin x



Graph of y = cos x

The functions $y = \sin x$ and $y = \cos x$ have the following characteristics.

1. The *domain* of each function is all real numbers.
2. The *range* of each function is $-1 \le y \le 1$.
3. Each function is **periodic**, which means that its graph has a repeating pattern that continues indefinitely. The shortest repeating portion is called a **cycle**. The horizontal length of each cycle is called the **period**. Each graph shown above has a period of 2π.
4. The maximum value of $y = \sin x$ is $M = 1$ and occurs when $x = \frac{\pi}{2} + 2n\pi$ where n is any integer. The maximum value of $y = \cos x$ is also $M = 1$ and occurs when $x = 2n\pi$ where n is any integer.
5. The minimum value of $y = \sin x$ is $m = -1$ and occurs when $x = \frac{3\pi}{2} + 2n\pi$ where n is any integer. The minimum value of $y = \cos x$ is also $m = -1$ and occurs when $x = (2n + 1)\pi$ where n is any integer.
6. The **amplitude** of each function's graph is $\frac{1}{2}(M - m) = 1$.

CHARACTERISTICS OF $Y = A \sin BX$ AND $Y = A \cos BX$

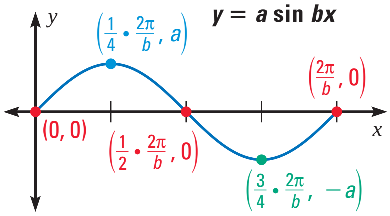
The amplitude and period of the graphs of $y = a \sin bx$ and $y = a \cos bx$, where a and b are nonzero real numbers, are as follows:

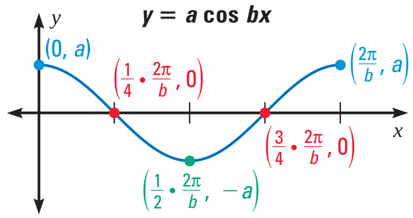
$$\text{amplitude} = |a| \qquad \text{and} \qquad \text{period} = \frac{2\pi}{|b|}$$

Examples The graph of $y = 2 \sin 4x$ has amplitude 2 and period $\frac{2\pi}{4} = \frac{\pi}{2}$.

The graph of $y = \frac{1}{3} \cos 2\pi x$ has amplitude $\frac{1}{3}$ and period $\frac{2\pi}{2\pi} = 1$.

For $a > 0$ and $b > 0$, the graphs of $y = a \sin bx$ and $y = a \cos bx$ each have five key x-values on the interval $0 \le x \le \frac{2\pi}{b}$: the x-values at which the **maximum** and **minimum** values occur and the ***x*-intercepts**.





EXAMPLE 1 *Graphing Sine and Cosine Functions*

Graph the function.

a. $y = 2 \sin x$ **b.** $y = \cos 2x$

SOLUTION

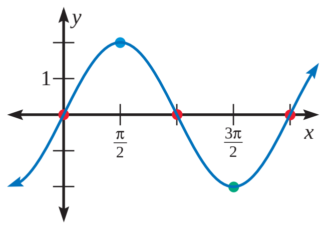
a. The amplitude is $a = 2$ and the period is $\frac{2\pi}{b} = \frac{2\pi}{1} = 2\pi$. The five key points are:

Intercepts: $(0, 0)$; $(2\pi, 0)$;

$\left(\frac{1}{2} \cdot 2\pi, 0\right) = (\pi, 0)$

Maximum: $\left(\frac{1}{4} \cdot 2\pi, 2\right) = \left(\frac{\pi}{2}, 2\right)$

Minimum: $\left(\frac{3}{4} \cdot 2\pi, -2\right) = \left(\frac{3\pi}{2}, -2\right)$



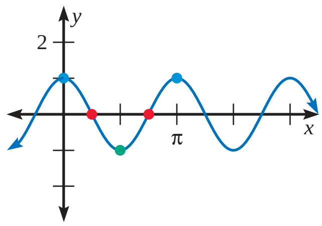
b. The amplitude is $a = 1$ and the period is $\frac{2\pi}{b} = \frac{2\pi}{2} = \pi$. The five key points are:

Intercepts: $\left(\frac{1}{4} \cdot \pi, 0\right) = \left(\frac{\pi}{4}, 0\right)$;

$\left(\frac{3}{4} \cdot \pi, 0\right) = \left(\frac{3\pi}{4}, 0\right)$

Maximums: $(0, 1)$; $(\pi, 1)$

Minimum: $\left(\frac{1}{2} \cdot \pi, -1\right) = \left(\frac{\pi}{2}, -1\right)$



STUDENT HELP

Study Tip
In Example 1 notice how changes in a and b affect the graphs of $y = a \sin bx$ and $y = a \cos bx$. When the value of a increases, the amplitude is greater. When the value of b increases, the period is shorter.

EXAMPLE 2 *Graphing a Cosine Function*

Graph $y = \frac{1}{3}\cos \pi x$.

SOLUTION

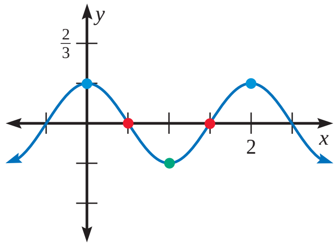
The amplitude is $a = \frac{1}{3}$ and the period is $\frac{2\pi}{b} = \frac{2\pi}{\pi} = 2$. The five key points are:

Intercepts: $\left(\frac{1}{4} \cdot 2, 0\right) = \left(\frac{1}{2}, 0\right)$;

$\left(\frac{3}{4} \cdot 2, 0\right) = \left(\frac{3}{2}, 0\right)$

Maximums: $\left(0, \frac{1}{3}\right)$; $\left(2, \frac{1}{3}\right)$

Minimum: $\left(\frac{1}{2} \cdot 2, -\frac{1}{3}\right) = \left(1, -\frac{1}{3}\right)$



The periodic nature of trigonometric functions is useful for modeling *oscillating* motions or repeating patterns that occur in real life. Some examples are sound waves, the motion of a pendulum or a spring, and seasons of the year. In such applications, the reciprocal of the period is called the **frequency**. The frequency gives the number of cycles per unit of time.

EXAMPLE 3 *Modeling with a Sine Function*

MUSIC When you strike a tuning fork, the vibrations cause changes in the pressure of the surrounding air. A middle-A tuning fork vibrates with frequency $f = 440$ hertz (cycles per second). You strike a middle-A tuning fork with a force that produces a maximum pressure of 5 pascals.

a. Write a sine model that gives the pressure P as a function of time t (in seconds).

b. Graph the model.

SOLUTION

a. In the model $P = a \sin bt$, the maximum pressure P is 5, so $a = 5$. You can use the frequency to find the value of b.

$$\text{frequency} = \frac{1}{\text{period}} \quad \rightarrow \quad 440 = \frac{b}{2\pi}$$

$$880\pi = b$$

▶ The pressure as a function of time is given by $P = 5 \sin 880\pi t$.

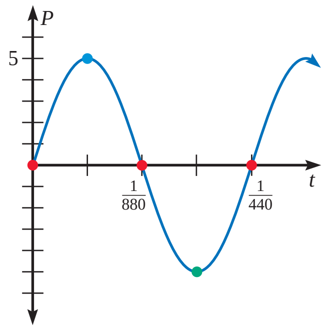
b. The amplitude is $a = 5$ and the period is $\frac{1}{f} = \frac{1}{440}$. The five key points are:

Intercepts: $(0, 0)$; $\left(\frac{1}{440}, 0\right)$;

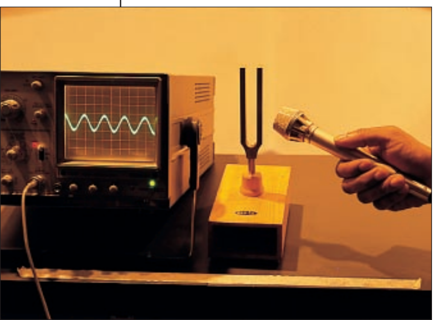
$\left(\frac{1}{2} \cdot \frac{1}{440}, 0\right) = \left(\frac{1}{880}, 0\right)$

Maximum: $\left(\frac{1}{4} \cdot \frac{1}{440}, 5\right) = \left(\frac{1}{1760}, 5\right)$

Minimum: $\left(\frac{3}{4} \cdot \frac{1}{440}, -5\right) = \left(\frac{3}{1760}, -5\right)$



FOCUS ON APPLICATIONS



REAL LIFE OSCILLOSCOPE The oscilloscope is a laboratory device invented by Karl Braun in 1897. This electrical instrument measures waveforms and is the forerunner of today's television.

GOAL 2 Graphing Tangent Functions

The graph of $y = \tan x$ has the following characteristics.

1. The domain is all real numbers except odd multiples of $\frac{\pi}{2}$. At odd multiples of $\frac{\pi}{2}$, the graph has vertical asymptotes.
2. The range is all real numbers.
3. The graph has a period of π.

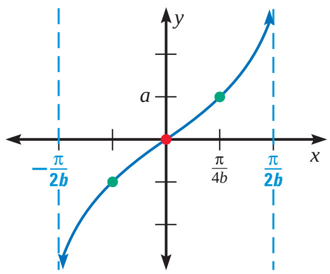
CHARACTERISTICS OF $Y = A \tan BX$

If a and b are nonzero real numbers, the graph of $y = a \tan bx$ has these characteristics:

- The period is $\frac{\pi}{|b|}$.
- There are vertical asymptotes at odd multiples of $\frac{\pi}{2|b|}$.

Example The graph of $y = 5 \tan 3x$ has period $\frac{\pi}{3}$ and asymptotes at $x = (2n + 1)\frac{\pi}{2(3)} = \frac{\pi}{6} + \frac{n\pi}{3}$ where n is any integer.

The graph at the right shows five key x-values that can help you sketch the graph of $y = a \tan bx$ for $a > 0$ and $b > 0$. These are the **x-intercept**, the x-values where the **asymptotes** occur, and the x-values **halfway between** the x-intercept and the asymptotes. At each halfway point, the function's value is either a or $-a$.



EXAMPLE 4 *Graphing a Tangent Function*

Graph $y = \frac{3}{2} \tan 4x$.

SOLUTION

The period is $\frac{\pi}{b} = \frac{\pi}{4}$.

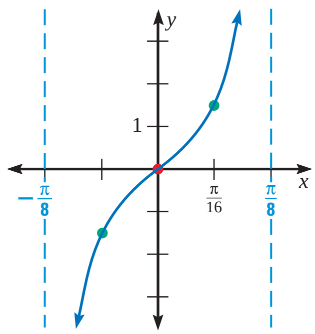
Intercept: $(0, 0)$

Asymptotes: $x = \frac{1}{2} \cdot \frac{\pi}{4}$, or $x = \frac{\pi}{8}$;

$x = -\frac{1}{2} \cdot \frac{\pi}{4}$, or $x = -\frac{\pi}{8}$

Halfway points: $\left(\frac{1}{4} \cdot \frac{\pi}{4}, \frac{3}{2}\right) = \left(\frac{\pi}{16}, \frac{3}{2}\right)$;

$\left(-\frac{1}{4} \cdot \frac{\pi}{4}, -\frac{3}{2}\right) = \left(-\frac{\pi}{16}, -\frac{3}{2}\right)$



GUIDED PRACTICE

Vocabulary Check ✓

1. Define the terms cycle and period.

Concept Check ✓

2. What are the domain and range of $y = a \sin bx$, $y = a \cos bx$, and $y = a \tan bx$?

3. Consider the two functions $y = 4 \sin \frac{x}{3}$ and $y = \frac{1}{3} \sin 4x$. Which function has the greater amplitude? Which function has the longer period?

Skill Check ✓

Find the amplitude and period of the function.

4. $y = 6 \sin x$
5. $y = 3 \cos \pi x$
6. $y = \frac{1}{4} \cos 3x$
7. $y = \frac{2}{3} \sin \frac{\pi}{3}x$
8. $y = 5 \sin 3\pi x$
9. $y = \cos \frac{x}{2}$

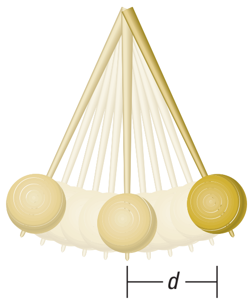
Graph the function.

10. $y = 3 \sin x$
11. $y = \cos 4x$
12. $y = \tan 3x$
13. $y = \frac{1}{4} \sin \pi x$
14. $y = 5 \cos \frac{2}{3}x$
15. $y = 2 \tan 4x$

16. **PENDULUMS** The motion of a certain pendulum can be modeled by the function

$$d = 4 \cos 8\pi t$$

where d is the pendulum's horizontal displacement (in inches) relative to its position at rest and t is the time (in seconds). Graph the function. How far horizontally does the pendulum travel from its original position?



PRACTICE AND APPLICATIONS

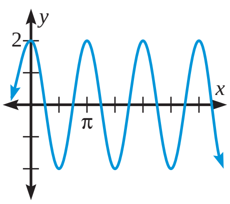
STUDENT HELP

Extra Practice to help you master skills is on p. 959.

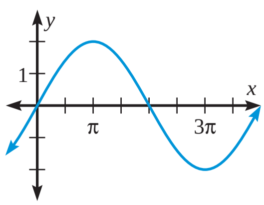
MATCHING GRAPHS **Match the function with its graph.**

17. $y = 2 \sin \frac{1}{2}x$
18. $y = 2 \cos \frac{1}{2}x$
19. $y = 2 \sin 2x$
20. $y = 2 \tan \frac{1}{2}x$
21. $y = 2 \cos 2x$
22. $y = 2 \tan 2x$

A.



B.



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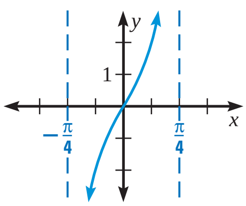
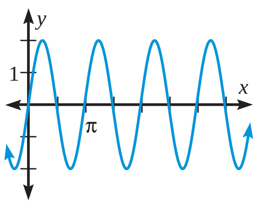
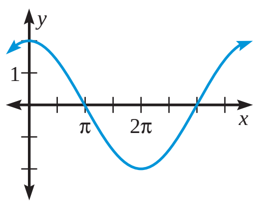


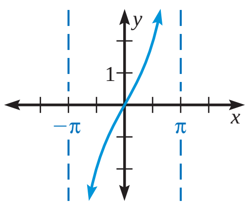
D.



E.



F.



STUDENT HELP

HOMEWORK HELP

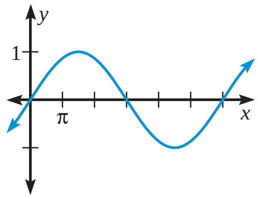
Examples 1, 2: Exs. 17–49
Example 3: Exs. 51–55
Example 4: Exs. 17–22, 32–43

STUDENT HELP

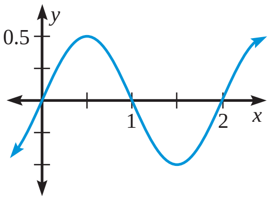
INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with problem solving in Exs. 44–49.

ANALYZING FUNCTIONS **In Exercises 23–31, find the amplitude and period of the graph of the function.**

23.



24.



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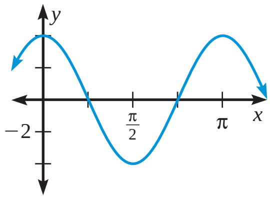


26. $y = \frac{1}{2}\cos \pi x$

27. $y = \sin 2x$

28. $y = 3\cos \frac{1}{4}x$

29. $y = 5\cos \frac{1}{2}x$

30. $y = 2\sin \frac{1}{2}\pi x$

31. $y = \frac{1}{3}\sin 4\pi x$

GRAPHING **Draw one cycle of the function's graph.**

32. $y = \sin \frac{1}{4}x$

33. $y = \cos \frac{1}{5}x$

34. $y = \frac{1}{4}\tan \pi x$

35. $y = \frac{1}{4}\sin x$

36. $y = 4\cos x$

37. $y = 4\tan 2x$

38. $y = 3\cos 2x$

39. $y = 8\sin x$

40. $y = 2\tan \frac{1}{3}x$

41. $y = \frac{1}{2}\sin \frac{1}{4}\pi x$

42. $y = \tan 4\pi x$

43. $y = 2\cos 6\pi x$

WRITING EQUATIONS **Write an equation of the form $y = a \sin bx$, where $a > 0$ and $b > 0$, so that the graph has the given amplitude and period.**

44. Amplitude: 1
Period: 5

45. Amplitude: 10
Period: 4

46. Amplitude: 2
Period: 2π

47. Amplitude: $\frac{1}{2}$
Period: 3π

48. Amplitude: 4
Period: $\frac{\pi}{6}$

49. Amplitude: 3
Period: $\frac{1}{2}$

FOCUS ON CAREERS

REAL LIFE MUSICIAN
Musicians may specialize in classical, rock, jazz, or many other types of music. This profession is very competitive and demands a high degree of discipline and talent in order to succeed.

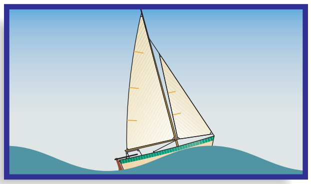
INTERNET CAREER LINK
www.mcdougallittell.com

50. LOGICAL REASONING Use the fact that the frequency of a periodic function's graph is the reciprocal of the period to show that an oscillating motion with maximum displacement a and frequency f can be modeled by $y = a \sin 2\pi ft$ or $y = a \cos 2\pi ft$.

51. BOATING The displacement d (in feet) of a boat's water line above sea level as it moves over waves can be modeled by the function

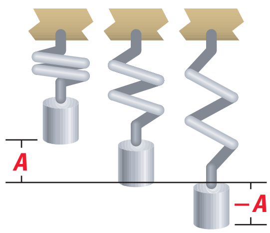
$$d = 2 \sin 2\pi t$$

where t is the time (in seconds). Graph the height of the boat over a three second time interval.

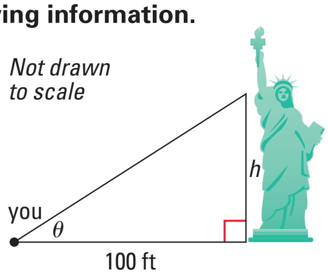


52. MUSIC A tuning fork vibrates with a frequency of 220 hertz (cycles per second). You strike the tuning fork with a force that produces a maximum pressure of 3 pascals. Write a sine model that gives the pressure P as a function of the time t (in seconds). What is the period of the sound wave?

53. SPRING MOTION The motion of a simple spring can be modeled by $y = A \cos kt$ where y is the spring's vertical displacement (in feet) relative to its position at rest, A is the initial displacement (in feet), k is a constant that measures the elasticity of the spring, and t is the time (in seconds). Find the amplitude and period of a spring for which $A = 0.5$ foot and $k = 6$.



SIGHTSEEING In Exercises 54 and 55, use the following information. Suppose you are standing 100 feet away from the base of the Statue of Liberty with a video camera. As you videotape the statue, you pan up the side of the statue at 5° per second.

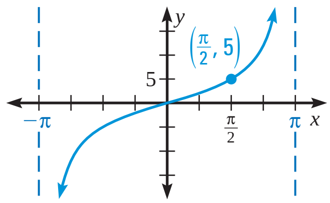


54. Write and graph an equation that gives the height h of the part of the statue seen through the video camera as a function of the time t.

55. Find the change in height from $t = 0$ to $t = 1$, from $t = 1$ to $t = 2$, and from $t = 2$ to $t = 3$. Briefly explain what happens to h as t increases.

Test Preparation

56. MULTIPLE CHOICE Which function represents the graph shown?



Ⓐ $y = \frac{1}{2} \tan 5x$ Ⓑ $y = 5 \tan \frac{1}{2}x$

Ⓒ $y = \tan 5x$ Ⓓ $y = 5 \tan 2x$

Ⓔ $y = 5 \tan \frac{1}{8}x$

57. MULTIPLE CHOICE Which of the following is an x-intercept of the graph of $y = \frac{1}{3} \sin \frac{\pi}{4}x$?

Ⓐ 4 Ⓑ 2 Ⓒ −6 Ⓓ 1 Ⓔ 4π

★ Challenge

SKETCHING GRAPHS Sketch the graph of the function by plotting points. Then state the function's domain, range, and period.

58. $y = \csc x$ **59.** $y = \sec x$ **60.** $y = \cot x$

MIXED REVIEW

GRAPHING Graph the quadratic function. Label the vertex and axis of symmetry. **(Review 5.1 for 14.2)**

61. $y = 2(x - 5)^2 + 4$ **62.** $y = -(x - 3)^2 - 7$ **63.** $y = 4(x + 2)^2 - 1$

64. $y = -3(x + 1)^2 + 6$ **65.** $y = \frac{3}{4}(x - 1)^2 - 2$ **66.** $y = 10(x + 4)^2 + 3$

CALCULATING PROBABILITY Find the probability of drawing the given numbers if the integers 1 through 30 are placed in a hat and drawn randomly without replacement. **(Review 12.5)**

67. an even number, then an odd number **68.** the number 30, then an odd number

69. a multiple of 4, then an odd number **70.** the number 19, then the number 20

FINDING REFERENCE ANGLES Sketch the angle. Then find its reference angle. **(Review 13.3)**

71. 220° **72.** −155° **73.** 280° **74.** −510°

75. $\frac{35\pi}{3}$ **76.** $\frac{21\pi}{4}$ **77.** $-\frac{17\pi}{6}$ **78.** $-\frac{5\pi}{8}$

79. PERSONAL FINANCE You deposit $1000 in an account that pays 1.5% annual interest compounded continuously. How long will it take for the balance to double? **(Review 8.6)**