This week is a little different because we will be answering the question:

“HOW DO I SOLVE POLYNOMIALS THAT ARE NOT FACTORABLE?”

The real answer, for your generation, is that you punch it into a graphing calculator and find the roots using technology. Your parents (or grandparents) had to do a lot more work to solve polynomials. While many of the principles they used are obsolete, it is still good for you to know about them. Let’s learn about them while trying to find the roots of

$$f(x) = x^4 + 4x^3 - 8x^2 - 24x - 45.$$

The Fundamental Theorem of Algebra (FTA) and its Corollary

Polynomial functions have at least one solution. If f(x) is a polynomial of degree n, and if a zero of multiplicity m is counted m times, then f(x) has precisely n zeros.

The FTA is basically telling us if we are looking at a polynomial function, there is at least one complex root. Complex does not guarantee that you will need to use *i* because a real number is a complex number whose imaginary part is zero. More importantly, the corollary (a fancy word for a statement that is related to a previous statement) tells us that **the number of roots is equal to the degree**.

Applying the theorem and corollary to our example, we should know that our function has _______ roots.

Descartes’ Rule of Signs

Let f(x) be a polynomial function. The number of ***positive*** *real zeros of f is equal to the number of coefficient sign changes of* ***f(x)*** *or is less than this by an even number. The number of* ***negative*** *real zeros of f is equal to the number of coefficient sign changes of* ***f(–x)*** *or is less than this by an even number.*

Let’s analyze our function using Descartes’ Rule of Signs. The first two coefficients of *f*(*x*) are positive (1 and 4), then the next coefficient is negative (–8). That is one coefficient sign change. The rest of the coefficients are negative, so the sign does not change again. If we look at *f*(–*x*), we have $f(-x) = x^4 - 4x^3 - 8x^2 + 24x - 45$. There are ________ coefficient sign changes in *f*(–*x*). Combining these two pieces of information with Descartes’, we now know that our function has _______ positive real roots and ______________ negative real roots. Let’s make a table of the possible scenarios involving the roots at this point. Remember each row should add to 4 because we know from FTA that our function has 4 roots.

	+	–	Complex
Possibility #1			
Possibility #2			

Rational Root Theorem (RRT)

For a polynomial function $a_n x^n + \ldots + a_1 x + a_0$*, every rational root must take the form* $\frac{c}{d}$*, where c is a factor of the constant term* a_0 *and d is a factor of the leading coefficient* a_n.

To apply the RRT, we need to make factor lists of the constant term and leading coefficient. In our example, the constant term is –45 and the leading coefficient is 1.

Factors of 45:	
Factors of 1:	

Now you must make fractions out of all the possible combinations of factors of 45 and factors of 1. In this case it is easy, because there is only one factor that can go in the denominator. You also should put a ± in front of each potential root because the factors could be positive or negative. So all of the possible roots for our function are:

Summary: So from using these three principles, we have gained some important information about our function. We know it has 4 roots. We know it has only 1 positive real root, so if we find one, we shouldn't waste more time looking for another. And we also have a list of possible rational roots, so we have limited the amount of guesswork to be done. From here, we need to learn a skill to help us find our roots…

Synthetic Division

Synthetic division does two things for us at once, which is why it is better than just plugging values in and seeing if they make the function equal zero. Thing #1 is tell us if the value is a root. Thing #2 is that it divides a factor out of the function and tells you the quotient. An important note about synthetic division is that **it can only be used when dividing by linear factors**. Let's look at dividing our function by the factor *x* – 1.

1	1	4	–8	–24	–45
	1				

Step #1: The value that we are testing goes in the box in the top left corner. In this case, we are testing 1 because that is the value that makes *x* – 1 (what we are dividing by) equal 0. Write out the coefficients of the function in different columns along the top row. If a term is missing, write a 0 for the coefficient. Also, bring down the leading coefficient into the bottom row.

Step #2: Multiply the value in the bottom row with the test value (the one in the box). Write the product in the next column's second row.

Step #3: Add the values in the next column and write the sum in the bottom row.

Step #4: Repeat until the final column is complete.

When you finish, the synthetic division should look like

1	1	4	–8	–24	–45
		1	5	–3	–27
	1	5	–3	–27	–72

The last number, –72 is in its own box for a reason. It is the remainder of your division. If $x - 1$ was a factor of our polynomial (which would mean $x = 1$ was a root), the remainder should be 0. This is because of

Remainder Theorem
If a polynomial f(x) is divided by x – c, then the remainder equals f(c).

Let's try another example. This time divide our function by $x + 5$.

	1	4	–8	–24	–45
	1				

Ah! The remainder equals zero, which means that $x = -5$ is one of the roots to our function. The bottom row tells us another piece of important information. The numbers that are not in the remainder box are the coefficients of the quotient. What this is telling us is that

$$\frac{x^4 + 4x^3 - 8x^2 - 24x - 45}{x+5} = x^3 - x^2 - 3x - 9 \text{ or}$$
$$x^4 + 4x^3 - 8x^2 - 24x - 45 = (x+5)(x^3 - x^2 - 3x - 9)$$

so we have found one factor of our initial function, and now we just need to follow the same process to factor $x^3 - x^2 - 3x - 9$.

1. Use FTA, Descartes', RRT, and synthetic division to find a root of $x^3 - x^2 - 3x - 9$.

2. Use the quadratic formula to find the final two roots.

Use FTA, Descartes', and RRT to find the roots of the following functions.

3. $f(x) = 2x^3 + 4x^2 - 17x + 2$

4. $g(x) = x^3 + 9x^2 + 16x - 6$

5. $h(x) = 2x^4 + 7x^3 - 11x + 2$

6. $k(x) = x^4 - 15x^2 + 10x + 24$